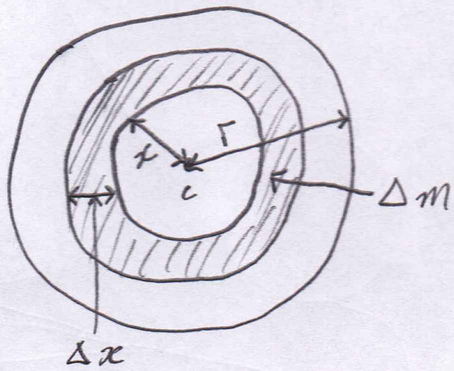
2013 Q8.

(a) Let c be the centre of the disc and ρ = mass per unit volume.

$$\rho = \frac{\text{mass}}{\text{Area}} \Rightarrow \text{Mass} = \rho(\text{Area})$$
$$\Rightarrow \text{Mass} = \rho\pi r^2$$

We break up the disc into infinitesimally thin hoops (Annuli) with centre c and each of width Δx where x = radius of the annulus.



Area of Annulus $= (2\pi x)(\Delta x)$

$$\Delta m = \rho(\text{Area})$$
$$= \rho\, 2\pi x \Delta x$$
$$= 2\pi\rho x \Delta x$$

$$I = \sum \Delta m\, r^2$$
$$= \sum (2\pi\rho x \Delta x) x^2$$
$$= \sum 2\pi\rho x^3 \Delta x$$

$$I = \int_0^r 2\pi\rho x^3\, dx$$
$$= \left[2\pi\rho \frac{x^4}{4}\right]_0^r$$
$$= 2\pi\rho\frac{r^4}{4} - 0$$
$$= \frac{1}{2}\pi\rho r^4$$

$$I = \frac{1}{2}(\pi\rho r^2) r^2$$
$$= \frac{1}{2} m r^2$$

since $m = \rho\pi r^2$

(b) (i) $$T = 2\pi\sqrt{\frac{I}{Mgh}}$$

$$= 2\pi\sqrt{\frac{\frac{1}{2}(8m)r^2 + (8m)r^2 + m(\sqrt{2}r)^2 + m(\sqrt{2}r)^2 + m(2r)^2}{8mg(r) + mg(r) + mg(r) + mg(2r)}}$$

$$= 2\pi\sqrt{\frac{20mr^2}{12mgr}}$$

$$= 2\pi\sqrt{\frac{5r}{3g}}$$

(ii) $$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{5r}{3g}}$$

$$\Rightarrow \frac{L}{g} = \frac{5r}{3g}$$

$$\Rightarrow L = \frac{5r}{3}.$$